Lecture 39

Sunday, December 5, 2021 12:36 AM

* Prayer
* Spiritual thought
* Nonhomogeneous system: $Y' = AY + F(t)$

Methods:

- Undetermined coefficients (when F is simple)
- Variation of parameters
- Laplace transform
- Decoupling (when A is diagonalizable)

We'll focus on the decoupling method and the variation of parameters.

- If A is a diagonal matrix, the system will be quite simple:

$$\begin{cases} y_1' = a_1 y_1 + f_1(t) \\ \vdots \\ y_n' = a_n y_n + f_n(t) \end{cases} \longrightarrow y_1, \dots, y_n \text{ can be solved independently.}$$

- If A is not a diagonal matrix, but is diagonalizable, then we can still decouple the system.

$A = PDP^{-1}$, where D is a diagonal matrix and P is an invertible matrix.

$$Y' = AY + F = PD\underbrace{P^{-1}Y}_{X} + F = PDX + F$$

$$\rightsquigarrow \underbrace{P^{-1}Y'}_{X'} = DX + P^{-1}F$$

$$\rightsquigarrow X' = DX + P^{-1}F$$

This is a decoupled system of equations $\to$ solvable.

Once we have X, we can find $Y = PX$.

Ex: $$Y' = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}}_{A} Y + \underbrace{\begin{bmatrix} t \\ e^t \end{bmatrix}}_{F}, \quad Y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find eigenvalues: $(\lambda-1)(\lambda-2) - 6 = \lambda^2 - 3\lambda - 4 = (\lambda+1)(\lambda-4)$.

$\lambda_1 = -1, \ \lambda_2 = 4$

Find eigenvectors:

$$\left.\begin{array}{l} A + I_2 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \rightsquigarrow v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\[2em] A - 4I_2 = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \rightsquigarrow v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{array}\right\} P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1}F = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} t \\ e^t \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3t - 2e^t \\ t + e^t \end{bmatrix}$$

$X = P^{-1}Y$

$X' = DX + P^{-1}F = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} X + \frac{1}{5}\begin{bmatrix} 3t-2e^t \\ t+e^t \end{bmatrix}$

$\rightsquigarrow \begin{cases} x_1' = -x_1 + \frac{1}{5}(3t-2e^t) \\ x_2' = 4x_2 + \frac{1}{5}(t+e^t) \end{cases}$

Solve for x_1, x_2 by integrating factor.

Initial cond:

$X(0) = P^{-1}Y(0) = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$

$Y = PX = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \cdots$

Variation of parameters:

$$Y' = AY + F \quad (*) \qquad \text{nonhomogeneous}$$

$$Y' = AY \qquad \text{homogeneous} \rightsquigarrow Y = \phi(t)\underbrace{C}_{\text{column vector of constants}}$$

To get a solution of $(*)$, we propose $Y = \phi(t)\underbrace{u(t)}_{\text{To be determined}}$.

$$Y' = \phi'(t)u(t) + \phi(t)u'(t) \overset{?}{=} AY + F = \underbrace{A\phi(t)}_{\phi'(t)}u(t) + F$$

$$\rightsquigarrow \quad \phi(t)u'(t) = F$$

$$\rightsquigarrow \quad u'(t) = \phi(t)^{-1}F = \frac{1}{5e^{3t}}\begin{bmatrix} e^{-t} & 2e^{4t} \\ -e^{-t} & 3e^{4t} \end{bmatrix}\begin{bmatrix} t \\ e^{t} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} te^{-4t} + 2e^{2t} \\ \cdots\cdots \end{bmatrix}$$

Then integrate to get $u(t)$:

$$u(t) = \begin{bmatrix} ? \\ ? \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Solution:

$$Y(t) = \Phi(t)u(t) = \frac{1}{5}\begin{bmatrix} e^{-4t} & 2e^{t} \\ -e^{-4t} & 3e^{t} \end{bmatrix}\left(\begin{bmatrix} ? \\ ? \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

Then plug $t=0$ to find c_1, c_2.